Primitive de la differentielle de f.

$$f(x,y) = \frac{x}{y}$$

2/

Exercice 1

Ecrivons les dérivée partielles jusqu'au deuxième ordre des fonctions suivantes.

1) $f(x,y) = \frac{\ln x}{x^2+y^2} \quad x \in \mathbb{R}_+^* \text{ et } y \in \mathbb{R}^*$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\frac{1}{x}(x^2+y^2) - 2x\ln x}{(x^2+y^2)^2}$$

$$= \frac{x + \frac{y^2}{x} - 2x\ln x}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{x(1-2\ln x) + \frac{y^2}{x}}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{-2y\ln x}{(x^2+y^2)^2}$$

* dérivée partielle secondaire.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x - 2x\ln x + \frac{y^2}{x}}{(x^2+y^2)^2}\right)$$

$$= 1 - \left(2\ln x + \frac{1}{x}2x\right) + \frac{-y^2}{x^2}$$

$$= \frac{\left[1 - 2\ln x - 2 - \frac{y^2}{x^2}\right](x^2+y^2)^2 - [4x(x^2+y^2)]\left(x - 2x\ln x + \frac{y^2}{x}\right)}{(x^2+y^2)^4}$$

$$= \frac{-1 - 2\ln x - \frac{y^2}{x^2}}{(x^2+y^2)^2} - \frac{4x\left(x - 2x\ln x + \frac{y^2}{x}\right)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{-1 - 2\ln x - \frac{y^2}{x^2}}{(x^2+y^2)^2} - \frac{4x\left(x - 2x\ln x + \frac{y^2}{x}\right)}{(x^2+y^2)^3}$$

Soit $b > 0$ [illegible]

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{-2y\ln x}{(x^2+y^2)^2}\right)$$

$$= \frac{-2\ln x\,(x^2+y^2)^2 - 4y(-2y\ln x)\cdot(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2\ln x}{(x^2+y^2)^2} + \frac{8y^2\ln x}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{x - 2x\ln x + \frac{y^2}{x}}{(x^2+y^2)^2}\right)$$

$$= \frac{\frac{2y}{x}(x^2+y^2)^2 - 4y\left(x - 2x\ln x + \frac{y^2}{x}\right)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\frac{2y}{x}}{(x^2+y^2)^2} - \frac{4y\left(x - 2x\ln x + \frac{y}{x}\right)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-2y\ln x}{(x^2+y^2)^2}\right)$$

$$= \frac{-\frac{1}{x}2y(x^2+y^2)^2 - 4x(-2y\ln x)}{(x^2+y^2)^4}$$

$$= \frac{-\frac{2y}{x}}{(x^2+y^2)^2} -$$

2) $g(x,y) = x\cos(x^2+xy)$

* Dérivée partielle première

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(x\cos(x^2+xy))$$

$$= \cos(x^2+xy) - x(2x+y)\sin(x^2+xy)$$

$$\boxed{\frac{\partial f}{\partial x}(x,y) = \cos(x^2+xy) - (2x^2+xy)\sin(x^2+xy)}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(x\cos(x^2+xy))$$

$$\frac{\partial f}{\partial y}(x,y) = -x \times x\ \sin(x^2+xy)$$

$$\boxed{\frac{\partial f}{\partial y}(x,y) = -x^2\sin(x^2+xy)}$$

* Dérivée partielle secondaire

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(\cos(x^2+xy) - (2x^2+xy)\sin(x^2+xy)\right)$$

$$= -(2x+y)\sin(x^2+xy) - \left[(4x+y)\sin(x^2+xy) + (2x+y)\cos(x^2+xy)(2x^2+xy)\right]$$

$$= -(2x+y)\sin(x^2+xy) - (4x+y)\sin(x^2+xy) - (4x^3+2x^2y+2x^2y+xy^2)\cos(x^2+xy)$$

$$= \sin(x^2+xy)(-2x-y-4x-y) - (4x^3+4x^2y+xy^2)\cos(x^2+xy)$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = (-6y-2y)\sin(x^2+xy) - (4x^3+4x^2y+xy^2)\cos(x^2+xy)}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(-x^2\sin(x^2+xy)\right)$$

$$\frac{\partial^2 f}{\partial y^2} = x\cdot(-x^2)\cos(x^2+xy)$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = -x^3\cos(x^2+xy)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(-x^2 \sin(x^2+xy)\right)$$

$$= -2x \sin(x^2+xy) + (2x+y)\cos(x^2+xy)(-x^2)$$

$$= -2\sin(x^2+xy) - x^2(2x+y)\cos(x^2+xy)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = -2\sin(x^2+xy) - (2x^3+x^2y)\cos(x^2+xy)}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\cos(x^2+xy) - (2x^2+xy)\sin(x^2+xy)\right)$$

$$= -x\sin(x^2+xy) - \left[x\sin(x^2+xy) + (\cancel{2x+x}\ (2x^2+xy)\, x\cos(x^2+xy)\right.$$

$$= -x\sin(x^2+xy) - \left[x\sin(x^2+xy) + (2x^3+x^2y)\cos(x^2+xy)\right.$$

$$\cancel{= \sin(x^2+xy)(=}$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = -2x\sin(x^2+xy) - (2x^3+x^2y)\cos(x^2+xy)}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{2y\ln x}{(x^2+y^2)^2}\right)$$

$$= \frac{-\frac{2y}{x}(x^2+y^2)^2 - 4x(x^2+y^2)(-2y\ln x)}{(x^2+y^2)^4}$$

$$= \frac{-\frac{2y}{x}(x^2+y^2)^2 + 8xy\ln x\,(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{2y}{x(x^2+y^2)^2} + \frac{8xy\ln x}{(x^2+y^2)^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{x - 2x\ln x + \frac{y^2}{x}}{(x^2+y^2)^2}\right)$$

$$= \frac{\frac{2y}{x}(x^2+y^2)^2 - 4y(x^2+y^2)\left(x - 2x\ln x + \frac{y^2}{x}\right)}{(x^2+y^2)^4}$$

$$\cancel{= \frac{\frac{2y}{x}}{(x^2+y^2)^2} - \frac{4y\left(x - 2x\ln x + \frac{y^2}{x}\right)}{(x^2+y^2)^3}}$$

$$\cancel{= \frac{\frac{2y}{x} - 4yx + 8xy\ln x - \frac{4y^3}{x}}{1}}$$

$$\cancel{\Rightarrow \frac{2y}{x(x^2+y^2)^2}}$$

$$\frac{(x^2+y^2)\left[\frac{2y}{x}(x^2+y^2) - 4y\left(x - 2x\ln x + \frac{y^2}{x}\right)\right]}{(x^2+y^2)^4}$$

$$= \frac{\frac{2y}{x}(x^2+y^2) - 4y\left(x - 2x\ln x + \frac{y^2}{x}\right)}{(x^2+y^2)^3}$$

$$= \frac{\frac{2y}{x}(x^2+y^2) - 4xy + 8xy\ln x - \frac{4y^3}{x}}{(x^2+y^2)^3}$$

$$= \frac{-\frac{2y}{x}\left[-(x^2+y^2) + 2x^2 + 2y^2\right] + 8xy\ln x}{(x^2+y^2)^3}$$

$$= \frac{-\frac{2y}{x}\left[-(x^2+y^2) + 2(x^2+y^2)\right] + 8xy\ln x}{(x^2+y^2)^3}$$

$$= \frac{-\frac{2y}{x}(x^2+y^2)}{(x^2+y^2)^3} + \frac{8xy\ln x}{(x^2+y^2)^3}$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{-2y}{x(x^2+y^2)^2} + \frac{8xy\ln x}{(x^2+y^2)^3}}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(-x^2 \sin(x^2+xy)\right)$$

$$= -2x \sin(x^2+xy) + (2x+y)(x^2+xy)\cos(x^2+xy)(-x^2)$$

$$= -2x\sin(x^2+xy) - x^2(2x+y)\cancel{(x^2+xy)}\cos(x^2+xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x\sin(x^2+xy) + (-2x^3-x^2y)\cancel{(x^2+xy)}\cos(x^2+xy)$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = -2x\sin(x^2+xy) + (-2x^3-x^2y)\cos(x^2+xy)}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\cos(x^2+xy) - (2x^2+xy)\sin(x^2+xy)\right)$$

$$= -x\sin(x^2+xy) - \left[x\sin(x^2+xy) + x(2x^2+xy)\cos(x^2+xy)\right]$$

$$= -x\sin(x^2+xy) - x\sin(x^2+xy) + (-2x^3-x^2y)\cos(x^2+xy)$$

$$\boxed{\frac{\partial^2 f}{\partial y \partial x} = -2x\sin(x^2+xy) + (-2x^3-x^2y)\cos(x^2+xy)}$$

Dérivée partielle première.

3) $h(x,y) = x^y \quad x \in \mathbb{R}_+^* \text{ et } y \in \mathbb{R}$

$$h(x,y) = e^{y\ln x}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(e^{y\ln x}\right)$$

$$= \frac{y}{x} e^{y\ln x}$$

$$\boxed{\frac{\partial f}{\partial x}(x,y) = \frac{y}{x} e^{y\ln x}}$$

$$\boxed{\frac{\partial f}{\partial y}(x,y) = \ln x \, e^{y\ln x}}$$

Dérivée partielle secondaire.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{y}{x} e^{y\ln x}\right)$$
$$= \frac{-y}{x^2} e^{y\ln x} + \frac{y}{x} \times \frac{y}{x} e^{y\ln x}$$
$$= \frac{y}{x^2} e^{y\ln x}(-1 + y)$$
$$= \frac{y}{x^2} e^{y\ln x}(y-1)$$
$$\frac{\partial^2 f}{\partial x^2} = (y-1)\frac{y}{x^2} e^{y\ln x}.$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right)$$
$$= \frac{\partial}{\partial y}\left(\ln x \, e^{y\ln x}\right)$$
$$= \ln x \, e^{y\ln x} . \ln x$$
$$\frac{\partial^2 f}{\partial y^2} = (\ln^2 x) e^{y\ln x}.$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right)$$
$$= \frac{\partial}{\partial x}\left(\ln x \, e^{y\ln x}\right)$$
$$= \frac{1}{x} e^{y\ln x} + \frac{y}{x}\ln x \, e^{y\ln x}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{x} e^{y\ln x}(1 + y\ln x)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right)$$
$$= \frac{\partial}{\partial y}\left(\frac{y}{x} e^{y\ln x}\right)$$
$$= \frac{1}{x} e^{y\ln x} + \frac{y}{x}\ln x \, e^{y\ln x}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{x} e^{y\ln x}(1 + y\ln x)$$

Exercice 2 OK

Montrons si les fonctions differentielles sont totale.

1. $\frac{1}{y}dx - \frac{x}{y^2}dy$

Posons $A(x,y) = \frac{1}{y}$ et

$B(x,y) = \frac{-x}{y^2}$

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y}\right)$$
$$\frac{\partial A(x,y)}{\partial y} = \frac{-1}{y^2}$$
$$\frac{\partial B(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-x}{y^2}\right)$$
$$= \cancel{\frac{\partial}{\partial x}(}$$
$$= \frac{-y^2}{y^4}$$
$$\frac{\partial B(x,y)}{\partial x} = \frac{-1}{y^2}$$

on a $\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}$

d'où $\frac{1}{y}dx - \frac{x}{y^2}dy$ est une fonction differentielle totale

2) $\cos(xy^2)dx + 2\cos(xy)dy$

Posons $A(x,y) = \cos(xy^2)$ et

$B(x,y) = 2\cos(xy)$

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial}{\partial y}(\cos(xy^2))$$
$$= -2xy\sin(xy^2)$$

$$\frac{\partial A(x,y)}{\partial x} = \frac{\partial}{\partial x}(2\cos(xy))$$
$$= -2y\sin(xy)$$

$$\frac{\partial A(x,y)}{\partial y} + \frac{\partial B(x,y)}{\partial x} \text{ d'où}$$

$\cos(xy^2)dx + 2\cos(xy)dy$ n'est pas une fonction différentielle totale.

Donnons les fonctions dont elles sont les différentielles

$$\int_x^y \frac{1}{y}\,dx = \frac{x}{y}$$

Exercice 3 OK

$$T = 2\pi\sqrt{\frac{l}{g}}$$

1) Calculons T pour un pendule de longueur $l = 1m$ avec $g = 9{,}807 m/s^2$

$$T = 2\pi\sqrt{\frac{1}{9{,}807}}$$

$T = 2\ s$

2) Calculons la variation de la période pour une petite variation de la longueur de 1cm

$T = 2\pi\sqrt{\frac{l}{g}}$ $l_f = l_i + 0{,}01 = 1 + 0{,}01$

$l_f = 1{,}02$

$$T = 2\pi\sqrt{\frac{1{,}01}{9{,}807}}$$

$T = 2{,}01\ s$

$\Delta T = T_f - T_i = 2{,}01 - 2$

~~$\Delta T = 0{,}805$~~ $\boxed{\Delta T = 0{,}011}$

Exercice 4

Soit $u = f(\sqrt{x^2+y^2})$

Trouvons les fonctions f telle que $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Déterminons $\frac{\partial^2 u}{\partial x^2}$ et $\frac{\partial^2 u}{\partial y^2}$

important $(f(ax))' = (ax)' \times f'(ax)$

$$\frac{\partial u}{\partial x} = (\sqrt{x^2+y^2})' \times f'(\sqrt{x^2+y^2})$$
$$= \frac{2x}{2\sqrt{x^2+y^2}} \times f'(\sqrt{x^2+y^2})$$
$$= \frac{x}{2\sqrt{x^2+y^2}} \times f'(\sqrt{x^2+y^2})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(x,y)\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) \times f'(\sqrt{x^2+y^2})$$
$$= \frac{\sqrt{x^2+y^2} - \frac{x \times 2x}{2\sqrt{x^2+y^2}}}{}$$

Exercice 3.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$T = 2\pi\frac{\sqrt{l}}{\sqrt{g}}$$

$$\Delta T = 2\pi\frac{1}{\sqrt{g}} \times \frac{\Delta l}{2\sqrt{l}}$$
$$= \frac{\pi}{\sqrt{gl}}\,\Delta l$$

$$\Delta T = \frac{\pi}{\sqrt{9{,}8 \times 1}} \times 10^{-2}$$

Exercice 1

Soit $u = f(\sqrt{x^2+y^2})$

Trouvons les fonctions f telle que $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Déterminons $\frac{\partial^2 u}{\partial x^2}$ et $\frac{\partial^2 y}{\partial y^2}$

$$\boxed{(f(ax))' = (ax)' \times f'(ax)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(f(\sqrt{x^2+y^2})\right)$$

$$= \frac{2x}{2\sqrt{x^2+y^2}} \times f'(\sqrt{x^2+y^2})$$

$$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \times f'(\sqrt{x^2+y^2})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(x,y)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \times f'(\sqrt{x^2+y^2})\right)$$

$$= \frac{\sqrt{x^2+y^2} - \frac{2x}{2\sqrt{x^2+y^2}} \times x}{(\sqrt{x^2+y^2})^2} f'(\sqrt{x^2+y^2}) + \frac{x}{\sqrt{x^2+y^2}} \times f''(\sqrt{x^2+y^2}) \times \frac{x}{\sqrt{x^2+y^2}}$$

$$\cancel{= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(\sqrt{x^2+y^2})^3} f'(\sqrt{x^2+y^2})^2 + \frac{x^2 f'(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2}}$$

$$= \frac{\frac{x^2+y^2-x^2}{\sqrt{x^2+y^2}}}{(\sqrt{x^2+y^2})^2} f'(\sqrt{x^2+y^2}) + \frac{x^2}{(\sqrt{x^2+y^2})^2} f''(\sqrt{x^2+y^2})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{(\sqrt{x^2+y^2})^3} f'(\sqrt{x^2+y^2}) + \frac{x^2}{x^2+y^2} f''(\sqrt{x^2+y^2})$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(f\left(\sqrt{x^2+y^2}\right)\right)$$

$$= \frac{2y}{2\sqrt{x^2+y^2}} \times f'\left(\sqrt{x^2+y^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \times f'\left(\sqrt{x^2+y^2}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}} \times f'\left(\sqrt{x^2+y^2}\right)\right)$$

$$= \frac{\sqrt{x^2+y^2} - \frac{2y}{2\sqrt{x^2+y^2}} \times y}{\left(\sqrt{x^2+y^2}\right)^2} f'\left(\sqrt{x^2+y^2}\right) + \frac{2y}{2\sqrt{x^2+y^2}} \times \frac{y}{\sqrt{x^2+y^2}} f''\left(\sqrt{x^2+y^2}\right)$$

$$= \frac{\frac{x^2+y^2-y^2}{\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2} f'\left(\sqrt{x^2+y^2}\right) + \frac{y^2}{x^2+y^2} f''\left(\sqrt{x^2+y^2}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{\left(\sqrt{x^2+y^2}\right)^3} f'\left(\sqrt{x^2+y^2}\right) + \frac{y^2}{x^2+y^2} f''\left(\sqrt{x^2+y^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \iff \left(\frac{x^2+y^2}{\left(\sqrt{x^2+y^2}\right)^3} f'\left(\sqrt{x^2+y^2}\right) + \frac{x^2+y^2}{x^2+y^2} f''\left(\sqrt{x^2+y^2}\right)\right) = 0$$

$$\frac{1}{\sqrt{x^2+y^2}} f' + f'' = 0 \iff \frac{1}{\sqrt{x^2+y^2}} f'\left(\sqrt{x^2+y^2}\right) + f''\left(\sqrt{x^2+y^2}\right) = 0$$

En posant $F = f'$ et $t = \sqrt{x^2+y^2}$ on a

$$\frac{1}{t} F + F' = 0 \iff \frac{1}{t} F = -F'$$

$$\iff \frac{F'}{F} = -\frac{1}{t} \iff \int \frac{F'}{F}\,dt = \int -\frac{1}{t}\,dt$$

$$\iff \ln|F| = -\ln|t| + c$$

$$\iff |F| = e^{-\ln|t| + c}$$

$$|F| = e^{-\ln|t|} e^{c}$$

$$F = \pm \frac{e^{c}}{|t|} \Leftrightarrow F(t) = \frac{k}{t} \text{ avec } k \in \mathbb{R}$$

$$F(t) = \frac{k}{t} \Leftrightarrow f'(t) = \frac{k}{t} \Leftrightarrow \int f'(t)\,dt = \int \frac{k}{t}\,dt$$

$\cancel{\ln|t|}$ $\Leftrightarrow \int f'(t)\,dt = \int \frac{k}{t}\,dt$

$$\Leftrightarrow f(t) = k \ln|t| + C_1$$

avec $k, C_1 \in \mathbb{R}$ et $t \in \mathbb{R}$

$$\Leftrightarrow \frac{\partial f}{\partial x} f(\sqrt{x^2+y^2}) = k \ln(\sqrt{x^2+y^2}) + C_1$$

$$(f(\sqrt{x^2+y^2}))' = (k \ln(\sqrt{x^2+y^2}) + C_1)'$$

$$= \frac{\frac{x}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y}} k = x$$

$$= \frac{x}{x^2+y^2} k = \frac{x}{\sqrt{x^2+y^2}}$$

$$k = \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

$$f(\sqrt{x^2+y^2}) = \frac{x^2+y^2}{\sqrt{x^2+y^2}} \ln(\sqrt{x^2+y^2})$$

$$= \sqrt{x^2+y^2} \ln(\sqrt{x^2+y^2}) + C_1 \quad C_1 \in \mathbb{R}$$

Exercice 5

$$PV = RT \Leftrightarrow P = \frac{RT}{V} \quad V = \frac{RT}{P} \quad T = \frac{PV}{R}$$

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$$

$$\frac{\partial P}{\partial V} = \frac{\partial}{\partial V}\left(\frac{RT}{V}\right)$$

$$\frac{\partial P}{\partial V} = \frac{-RT}{V^2}$$

$$\frac{\partial V}{\partial T} = \frac{\partial}{\partial T}\left(\frac{RT}{P}\right)$$

$$\frac{\partial V}{\partial T} = \frac{RP}{P^2} = \frac{R}{P}$$

$$\frac{\partial T}{\partial P} = \frac{\partial}{\partial P}\left(\frac{PV}{R}\right)$$

$$\frac{\partial T}{\partial P} = \frac{VR}{R^2} = \frac{V}{R}$$

$$\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = \frac{-RT}{V^2} \cdot \frac{R}{P} \cdot \frac{V}{R}$$

$$= \frac{-RT}{VP}$$

$$= \frac{-R \cdot \frac{PV}{R}}{VP}$$

$$= -1.$$

d'où $\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = -1.$

$$T \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = R$$

$$\frac{\partial P}{\partial T} = \frac{\partial}{\partial T}\left(\frac{RT}{V}\right)$$

$$= \frac{R}{V}$$

$$\frac{\partial V}{\partial T} = \frac{R}{P}$$

$$T \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = T \cdot \frac{R}{V} \cdot \frac{R}{P}$$

$$= \frac{PV}{R} \cdot \frac{R}{V} \cdot \frac{R}{P}$$

$$T \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = R$$

Exercice 6

1) Déterminons en tout point de $\mathbb{R}^2$ le vecteur gradient d'application

$f(x,y) \mapsto x e^{-(x^2+y^2)}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x e^{-(x^2+y^2)}\right)$$

$$= e^{-(x^2+y^2)} + -2x e^{-(x^2+y^2)} \times x$$

$$= e^{-(x^2+y^2)} - 2x^2 e^{-(x^2+y^2)}$$

$$\frac{\partial f}{\partial x} = e^{-(x^2+y^2)}\left(1-2x^2\right)$$

$$\frac{\partial f}{\partial x} = (1-2x^2)e^{-(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x e^{-(x^2+y^2)}\right)$$

$$= -2y e^{-(x^2+y^2)} \times x$$

$$\frac{\partial f}{\partial y} = -2xy e^{-(x^2+y^2)}$$

$$\nabla f = \begin{pmatrix} (1-2x^2)e^{-(x^2+y^2)} \\ -2xy e^{-(x^2+y^2)} \end{pmatrix}$$

2) $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(x,y,z) \longmapsto (e^{2y}+e^{2z}, e^{2x}-e^{2z}, x-y)$

Calculons le rotationnel de f en tout point de $\mathbb{R}^3$.

$f_1(x,y,z) = e^{2y}+e^{2z}$ $\quad f_3(x,y,z) = x-y$

$f_2(x,y,z) = e^{2x}+e^{2z}$

$$\mathrm{rot}(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}$$

$$\mathrm{rot}(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial y} f_3(x,y,z) - \frac{\partial}{\partial z} f_2(x,y,z) \\ \frac{\partial}{\partial z} f_1(x,y,z) - \frac{\partial}{\partial x} f_3(x,y,z) \\ \frac{\partial}{\partial x} f_2(x,y,z) - \frac{\partial}{\partial y} f_1(x,y,z) \end{pmatrix}$$

$$\frac{\partial}{\partial y} f_3(x,y,z) = \frac{\partial}{\partial y}(x-y)$$

$$= -1.$$

$$\frac{\partial}{\partial z} f_2(x,y,z) = \frac{\partial}{\partial z}(e^{2x}-e^{2z})$$

$$= -2e^{2z}.$$

$$\frac{\partial}{\partial z} f_1(x,y,z) = \frac{\partial}{\partial z}(e^{2y}+e^{2z})$$

$$= 2e^{2z}$$

$$\frac{\partial}{\partial x} f_3(x,y,z) = \frac{\partial}{\partial x}(x-y)$$

$$= 1$$

$$\frac{\partial}{\partial x} f_2(x,y,z) = \frac{\partial}{\partial x}(e^{2x}-e^{2z})$$

$$= 2e^{2x}$$

$$\frac{\partial f}{\partial y} f_1(x,y,z) = \frac{\partial}{\partial y}(e^{2y}+e^{2z})$$

$$= 2e^{2y}.$$

$$\mathrm{rot}(x,y,z) = \begin{pmatrix} -1 + 2e^{2z} \\ 2e^{2z} - 1 \\ 2(e^{2x} - e^{2y}) \end{pmatrix}$$

3) $\mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (e^x \cos(y), e^x \sin(y))$

Déterminons la divergence de f en tout point.

$\mathrm{div} = \nabla \cdot f(x,y)$

$f_1(x,y) = e^x \cos(y)$

$f_2(x,y) = e^x \sin(y)$

$\frac{\partial}{\partial x} f_1(x,y) = \frac{\partial}{\partial x}(e^x \cos(y))$

$= e^x \cos(y)$

$\frac{\partial}{\partial x} f_2(x,y) = \frac{\partial}{\partial x}(e^x \sin(y))$

$= e^x \sin(y)$

$\frac{\partial}{\partial y} f_1(x,y) = \frac{\partial}{\partial y}(e^x \cos(y))$

$= -\sin(y) e^x$

$= -e^x \sin(y)$

$\frac{\partial}{\partial y} f_2(x,y) = \frac{\partial}{\partial y}(e^x \sin(y))$

$= \cos(y) e^x$

$= e^x \cos(y)$

$$\mathrm{div} = \begin{pmatrix} e^x \cos(y) - e^x \sin(y) \\ e^x \sin(y) + e^x \cos(y) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\mathrm{div} = \begin{pmatrix} e^x(\cos(y) - \sin(y)) \cdot e^x \cos(y) \\ e^x(\sin(y) + \cos(y)) \cdot e^x \sin(y) \end{pmatrix}$$

$$\mathrm{div} = \begin{pmatrix} e^{2x} \cos(y) (\cos(y) - \sin(y)) \\ e^{2x} \sin(y) (\sin(y) + \cos(y)) \end{pmatrix}$$

4) $\mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = x^2 - xy^3 - y^2 z + z^3$

Déterminons le laplacien de f en tout point de $\mathbb{R}^3$

$\Delta f(x,y,z) \quad \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i}(x,y,z)$

$\Delta f = \frac{\partial^2}{\partial x^2}(x^2 - xy^3 - y^2 z + z^3)$

$+ \frac{\partial^2}{\partial z^2}(x^2 - xy^3 - y^2 z + z^3) +$

$\frac{\partial^2}{\partial y^2}(x^2 - xy^3 - y^2 z + z^3)$

Exercice 7

$F(r,\theta) = f(r\cos\theta, r\sin\theta)$

1) Montrons que F est de classe C^2 sur $\mathbb{R}_+ \times \mathbb{R}$

Les fonctions $(r,\theta) \to r\cos\theta$ et $(r,\theta) \mapsto r\sin\theta$ sont de classe

C^2 sur $\mathbb{R}^* \times \mathbb{R}$ en tant que produit de fonction de classe C^2 or f est C^2 sur $\mathbb{R}^2 \setminus \{0,0\}$
Par suite F est de classe C^2 en tant que composée de fonction de classe C^2

2/ Montrons que pour tout $(r,\theta) \in \mathbb{R}^*_+ \times \mathbb{R}$, on a

$$\Delta f(r\cos\theta, r\sin\theta) = \sum_{i=1}^{2} \frac{\partial^2}{\partial(r,\theta)^2}$$

~~$\Leftrightarrow \frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta) = (\cos\theta, \sin\theta) \times f^{-1}(r\cos\theta, r\sin\theta)$~~

$$\Leftrightarrow \Delta f(r,\theta) = \frac{\partial^2}{\partial r^2} F(r,\theta) + \frac{\partial^2}{\partial \theta^2} F(r,\theta)$$

$$\Leftrightarrow \frac{\partial f}{\partial r} f(r\cos\theta, r\sin\theta) = (\cos\theta, \sin\theta) \times f'(r\cos\theta, r\sin\theta)$$

$$\frac{\partial}{\partial r}(\cos\theta, \sin) \times f'(r\cos\theta, r\sin\theta)$$

Exercice 2

1/ Ecrivons les composantes des vecteur $\vec{OM}$

Dans le repère cartésien.

$$\vec{OM} = x\vec{e}_x + y\vec{e}_y$$

Dans le repère locale polaire

$$\vec{OM} = \rho\cos\varphi\,\vec{e}_x + \rho\sin\varphi\,\vec{e}_y$$

$$\vec{e}_\rho = \frac{\vec{OM}}{OM} \quad \text{or} \quad OM = \rho$$

$$\vec{e}_\rho = \frac{1}{\rho}(\rho\cos\varphi\,\vec{e}_x + \rho\sin\varphi\,\vec{e}_y)$$

$$\vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y$$

$$\vec{e}_{\varphi} = \frac{\partial \vec{OM}}{\partial \varphi}$$

$$= \frac{\partial(\cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y)}{\partial\varphi}$$

$$\vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y$$

$$\begin{cases} \vec{e}_\rho = \cos\varphi\,\vec{e}_x + \sin\varphi\,\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y \end{cases}$$

$\cos\varphi \Big\} \cos\varphi\cos\varphi\,\vec{e}_x + \cos\varphi\sin\varphi\,\vec{e}_y$

$-\sin\varphi \Big\} -\sin\varphi(-\sin\varphi)\vec{e}_x - \sin\varphi\cos\varphi\,\vec{e}_y$

$$\begin{cases} \cos\varphi\, \vec{e}_\rho = \cos\varphi \cos\varphi\, \vec{e}'_x + \cos\varphi \sin\varphi\, \vec{e}'_y \\ -\sin\varphi\, \vec{e}'_\varphi = \sin\varphi \sin\varphi\, \vec{e}'_x - \sin\varphi \cos\varphi\, \vec{e}'_y \end{cases}$$

$$\cos\varphi\, \vec{e}_\rho - \sin\varphi\, \vec{e}_\varphi = \vec{e}'_x + \cos\varphi \sin\varphi\, \vec{e}'_y - \sin\varphi \cos\varphi\, \vec{e}'_y$$

$$\vec{e}'_x = \cos\varphi\, \vec{e}'_\rho - \sin\varphi\, \vec{e}'_\varphi$$

$$\vec{e}'_y = \sin\varphi\, \vec{e}'_\rho + \cos\varphi\, \vec{e}'_\varphi$$

$$\vec{OM} = \rho\cos\varphi\,(\cos\varphi\, \vec{e}'_\rho - \sin\varphi\, \vec{e}'_\varphi) + \rho\sin\varphi\,(\sin\varphi\, \vec{e}'_\rho + \cos\varphi\, \vec{e}'_\varphi)$$

$$\vec{OM} = \rho\cos^2\varphi\, \vec{e}'_\rho - \rho\cos\varphi\sin\varphi\, \vec{e}'_\varphi + \rho\sin^2\varphi\, \vec{e}'_\rho + \rho\sin\varphi\cos\varphi\, \vec{e}'_\varphi$$

$$\boxed{\vec{OM} = \rho\, \vec{e}'_\rho}$$

2) Calculons $\frac{\partial \vec{OM}}{\partial \rho}$

$$\frac{\partial \vec{OM}}{\partial \rho} = \frac{\partial}{\partial \rho}(\rho\, \vec{e}'_\rho)$$

$$\boxed{\frac{\partial \vec{OM}}{\partial \rho} = \vec{e}'_\rho}$$

$$\frac{\partial \vec{OM}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(\rho\, \vec{e}'_\rho)$$

$$\frac{\partial \vec{OM}}{\partial \varphi} = \rho\,\frac{\partial}{\partial \varphi}(\vec{e}'_\rho)$$

$$\frac{\partial \vec{OM}}{\partial \varphi} = \rho\, \vec{e}_\varphi$$

car

$$\vec{e}_\rho = \cos\varphi\, \vec{e}'_x + \sin\varphi\, \vec{e}'_y$$

$$\vec{e}_\varphi = \frac{\partial \vec{OM}}{\partial \varphi} = -\sin\varphi\, \vec{e}'_x + \cos\varphi\, \vec{e}'_y$$

$$\frac{\partial \vec{e}_\rho}{\partial \varphi} = -\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y = \vec{e}_\varphi$$

Déduisons l'expression du déplacement.

$$d\vec{OM} = \frac{\partial \vec{OM}}{\partial \rho} d\rho + \frac{\partial \vec{OM}}{\partial \varphi} d\varphi$$

$$d\vec{OM} = \vec{e}_\rho\, d\rho + \rho\, \vec{e}_\varphi\, d\varphi$$

3) $\rho = a e^{\varphi}$

Calculons $d\vec{OM}$

$$d\vec{OM} = d\rho\, \vec{e}_\rho + \rho\, d\varphi\, \vec{e}_\varphi$$

$$d\vec{OM} = d(a e^{\varphi})\, \vec{e}_\rho + (a e^{\varphi})\, d\varphi\, \vec{e}_\varphi$$

$$= a e^{\varphi}\, d\varphi\, \vec{e}_\rho + a e^{\varphi}\, d\varphi\, \vec{e}_\varphi$$

$$\boxed{d\vec{OM} = a e^{\varphi}\, d\varphi\, (\vec{e}_\rho + \vec{e}_\varphi)}$$

4) $\varphi = \omega t$

Les composante du vecteur $\frac{d\vec{OM}}{dt}$ dans le repère polaire

$$\frac{d\vec{OM}}{dt} = \frac{d\rho}{dt}\vec{e}_\rho + \rho \frac{d\varphi}{dt}\vec{e}_\varphi$$

$$= \dot{\rho}\, \vec{e}_\rho + \rho\, \dot{\varphi}\, \vec{e}_\varphi$$

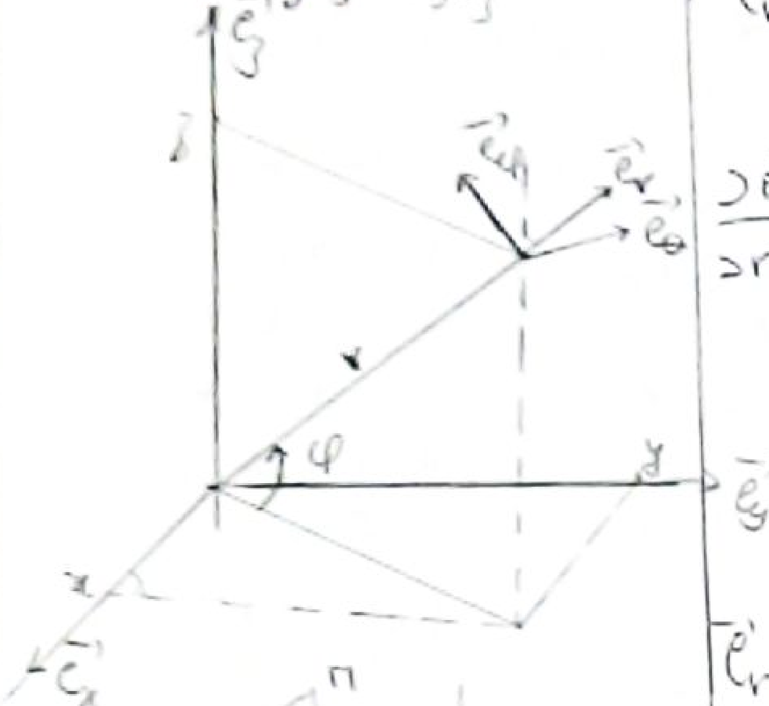
$$\frac{d\vec{OM}}{dt} = \dot{\rho}\, \vec{e}_\rho + \rho\, \omega\, \vec{e}_\varphi$$

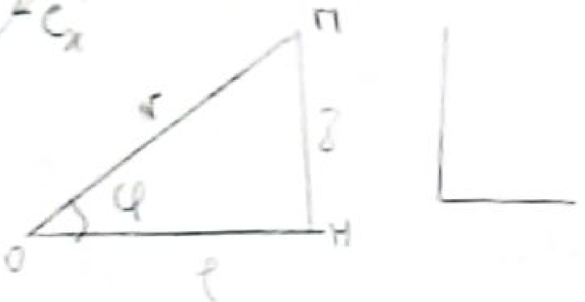
Exercices

$(O, \vec{e}_x ; \vec{e}_y , \vec{e}_z)$

1) Les composantes des vecteurs unitaires associé à M en coordonnées polaire

$\overrightarrow{OM} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$





En coordonnées cyli Cylindriq

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

Sphérique

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\sin\theta \\ z = r\sin\varphi \end{cases}$$

$\overrightarrow{OM} = r\cos\varphi\cos\theta\,\vec{e}_x + r\cos\varphi\sin\theta\,\vec{e}_y + r\sin\varphi\,\vec{e}_z$

$\vec{e}_r$; $\vec{e}_\theta$ et $\vec{e}_\varphi$

$$\vec{e}_r = \frac{\dfrac{\partial \overrightarrow{OM}}{\partial r}}{\left\| \dfrac{\partial \overrightarrow{OM}}{\partial r} \right\|}$$

$$\frac{\partial \overrightarrow{OM}}{\partial r} = \cos\varphi\cos\theta\,\vec{e}_x + \cos\varphi\sin\theta\,\vec{e}_y + \sin\varphi\,\vec{e}_z$$

$$\left\| \frac{\partial \overrightarrow{OM}}{\partial r} \right\| = \sqrt{\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi} = 1$$

$\vec{e}_r = \cos\varphi\cos\theta\,\vec{e}_x + \cos\varphi\sin\theta\,\vec{e}_y + \sin\varphi\,\vec{e}_z$

$$\vec{e}_\theta = \frac{\dfrac{\partial \overrightarrow{OM}}{\partial \theta}}{\left\| \dfrac{\partial \overrightarrow{OM}}{\partial \theta} \right\|}$$

$$\frac{\partial \overrightarrow{OM}}{\partial \theta} = -r\cos\varphi\sin\theta\,\vec{e}_x + r\cos\varphi\cos\theta\,\vec{e}_y$$

$$\left\| \frac{\partial \overrightarrow{OM}}{\partial \theta} \right\| = \sqrt{(-r\cos\varphi\sin\theta)^2 + (r\cos\varphi\cos\theta)^2}$$

$$= \sqrt{r^2\cos^2\varphi\sin^2\theta + r^2\cos^2\varphi\cos^2\theta}$$

$$= \sqrt{r^2\cos^2\varphi(\underbrace{\sin^2\theta + \cos^2\theta}_{1})}$$

$$\left\| \frac{\partial \overrightarrow{OM}}{\partial \theta} \right\| = r\cos\varphi$$

$\vec{e}_\theta = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y$

$$\vec{e}_\varphi = \frac{\frac{\partial \overrightarrow{OM}}{\partial \varphi}}{\left\| \frac{\partial \overrightarrow{OM}}{\varphi} \right\|}$$

$$\frac{\partial \overrightarrow{OM}}{\partial \varphi} = -r\sin\varphi\cos\theta\,\vec{e}_x - r\sin\varphi\sin\theta\, e_y + r\cos\varphi\,\vec{e}_3$$

$$\left\| \frac{\partial \overrightarrow{OM}}{\varphi} \right\| = \sqrt{r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi}$$

$$= \sqrt{\underbrace{r^2\sin^2\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) + r^2\cos^2\varphi}_{1}}$$

$$= \sqrt{r^2}$$

$$\left\| \frac{\partial \overrightarrow{OM}}{\varphi} \right\| = r$$

$$\vec{e}_\varphi = -\sin\varphi\cos\theta\, e$$